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γ_i-deformed Lax pair for rotating strings in the fast motion limit

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Abstract: A 3-parameter generalization of the Lunin-Maldacena background has recently been constructed by Frolov. This γ_i-deformed background is non-supersymmetric. We consider strings in this γ_i-deformed $\mathbb{R}\times S^5$ background rotating in three orthogonal planes (the 3-spin sector) in a fast motion limit, in which the total angular momentum J is assumed to be large. We show that there exists a consistent transformation which takes the undeformed equations of motion into the γ_i-deformed equations of motion. This transformation is used to construct a Lax pair for the bosonic part of the γ_i-deformed theory in the fast motion limit. This implies the integrability of the bosonic part of the γ_i-deformed string sigma model in the fast motion limit.

Keywords: AdS-CFT Correspondence, Integrable Equations in Physics.

Contents

1. Introduction

The original conjecture of Maldacena [1], which was further elaborated in [2, 3], claims a correspondence between string theory in an $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. It is well known [4] that one can construct a marginal deformation of $\mathcal{N} = 4$ SYM theory to obtain an $\mathcal{N} = 1$ superconformal SYM theory. These include the so-called β-deformations. Lunin and Maldacena have found gravity backgrounds which are conjectured to be dual to these β-deformations [5]. Evidence for this conjecture includes the matching of the energies of semi-classical strings to the anomalous dimensions of corresponding operators in the gauge theory [6] and a study of the pp-wave limit of the string theory together with an identification of the dual BMN operators [7−9]. Furthermore, the central charge was reproduced in the dual gravitational description and shown to be independent of the deformation parameter [10, 11]. This verifies on the string theory side that β-deformations are indeed marginal. Frolov was able to find a Lax pair representation for the β-deformed string theory in the case of a real deformation parameter $\beta = \gamma$ [12]. The crucial insight in this work was that the Lunin-Maldacena deformation, in the case of a real deformation parameter γ, can be realized as a TsT transformation of the string action with shift parameter $\tilde{\gamma} = \sqrt{\lambda}\gamma$, where $\sqrt{\lambda} = R^2$ and R is the radius of S^5.

Frolov also constructed a non-supersymmetric γ_i-deformed string theory by performing a series of three TsT transformations with shift parameters $\tilde{\gamma}_i = \sqrt{\lambda}\gamma_i$ on the string action and conjectured a correspondence to a non-supersymmetric γ_i-deformed SYM theory [12].[1] Remarkably, this 3-parameter deformation also admits a Lax pair representation. Evidence for this proposed correspondence was given by Frolov, Roiban and Tseytlin [15], who matched the semi-classical energies of strings to the anomalous dimensions of gauge theory operators using a γ_i-deformed spin chain representation. Furthermore, agreement was also found between the string and gauge theory descriptions of open strings attached to unstable giant gravitons [16]. These comparisons are particularly interesting because any agreement found cannot be the result of supersymmetry on either side, since these γ_i-deformed string and gauge theories are non-supersymmetric. The special case of the γ_i-deformed background with equal shift parameters $\gamma_i = \gamma$ is the same as the β-deformed background with real $\beta = \gamma$.

The conjectured gauge/string theory duality is of the strong-weak coupling type with respect to the 't Hooft coupling. This makes comparison between the two sides difficult. In this regard, studying a semi-classical limit of the theory is surprisingly fruitful as it allows perturbative computations in the gauge theory to be matched to classical computations in the dual gravitational theory [17, 18].

The semi-classical limit formulated and used in [6, 15, 19 − 24], which will henceforth be known as the fast motion limit, will now be described in detail. We consider strings in $\mathbb{R} \times S^5$ rotating in three orthogonal planes (the 3-spin sector)[2] and assume that the total angular momentum J is large. The string action in the fast motion limit is then obtained by isolating the "fast" angular coordinate (which corresponds to the total angular momentum), choosing a suitable gauge and assuming that the time derivatives of the radii and other "slow" angular coordinates are small (of order $\tilde{\lambda} = \frac{\lambda}{J^2}$). The gauge generally used is known as the non-diagonal uniform gauge and ensures that the angular momentum is spread evenly along the string (this makes comparison to the effective action of the corresponding spin chain on the gauge theory side possible).

The lagrangian governing the dynamics of strings rotating in three orthogonal planes in the γ_i-deformed $\mathbb{R}\times S^5$ background in the fast motion limit was derived in [15] and shown to be equivalent on the gauge theory side to the effective action of a γ_i-deformed spin chain in the continuum limit (where the length of the spin chain J becomes large). Now, for the case of a similar lagrangian in the undeformed background (which was originally derived in [22 − 24], but was also obtained in [15] by setting γ_i=0), the undeformed equations of

[1] For further discussions of backgrounds arising from TsT transformations see [13, 14].

[2] Multispin solutions describing rotating strings in the Lunin-Maldacena background were found in [25].

JHEP01(2006)050

motion are equivalent to a Landau-Lifshitz equation for which there is a known Lax pair representation [15, 23]. It is not known, however, whether the dynamics following from the γ_i-deformed lagrangian are integrable. In this article we will construct a Lax pair representation for the γ_i-deformed equations of motion, thereby settling this issue.

In section 2 we review the undeformed case. The undeformed equations of motion in the fast motion limit are derived and shown to be equivalent to the zero curvature condition for the undeformed Lax pair. In section 3 we derive the γ_i-deformed equations of motion and construct a transformation which takes the undeformed equations of motion into these γ_i-deformed equations of motion. We then choose a suitable gauge for the undeformed Lax pair and use this transformation to derive a γ_i-deformed Lax pair. A brief conclusion is then given in section 4 and some of the more lengthy calculations are included in appendices A, B and C.

2. Strings in the undeformed background

2.1 Equations of motion in the undeformed background

Let us consider strings in the undeformed $\mathbb{R} \times S^5$ background (at the center of AdS_5) rotating in three orthogonal planes. The total angular momentum is $J = J_1 + J_2 + J_3$, where J_i is the angular momentum associated with the i^{th} angular coordinate $\tilde{\tilde{\phi}}_i$ (where i=1, 2, 3).[3] In the fast motion limit, where the total angular momentum J is large and the time derivatives of the radii and "slow" angular coordinates are assumed to be of order $\tilde{\lambda} = \frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2}$, the string action to first order in $\tilde{\lambda}$ (derived in [15]) is

$$S = J \int d\tau \frac{d\sigma}{2\pi} [\mathcal{L} + O(\tilde{\lambda}^2)] , \tag{2.1}$$

where the lagrangian in the undeformed background is[4]

$$\mathcal{L} = \sum_{i=1}^{3} r_i^2 \dot{\tilde{\tilde{\phi}}}_i - \frac{\tilde{\lambda}}{2} \left\{ \sum_{i=1}^{3} (r_i')^2 + \sum_{\substack{i,j=1 \\ i<j}}^{3} r_i^2 r_j^2 (\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_j')^2 + \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) \right\} . \tag{2.2}$$

The last term is a constraint term. Re-defining $\tau \to \frac{1}{\tilde{\lambda}} \tau$ and $\mathcal{L} \to -\frac{1}{\tilde{\lambda}} \mathcal{L}$ gives

$$\mathcal{L} = -\sum_{i=1}^{3} r_i^2 \dot{\tilde{\tilde{\phi}}}_i + \frac{1}{2} \sum_{i=1}^{3} (r_i')^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i<j}}^{3} r_i^2 r_j^2 \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_j' \right)^2 + \frac{1}{2} \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) . \tag{2.3}$$

The equations of motion, obtained by varying with respect to r_i and $\tilde{\tilde{\phi}}_i$ respectively, are

$$r_i'' = -2 r_i \dot{\tilde{\tilde{\phi}}}_i + r_i \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k' \right)^2 + \Lambda r_i , \tag{2.4}$$

[3] We make use of the notation of [12] for the angular coordinates in the undeformed and γ_i-deformed backgrounds.

[4] It should be noted that henceforth the Einstein summation convention will not be used. All summations will be explicitly mentioned so as to avoid confusion.

JHEP01(2006)050

$$\dot{r}_i = \sum_{k=1}^{3} r_k (r_i r_k)' \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k' \right) + \frac{1}{2} r_i \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i'' - \tilde{\tilde{\phi}}_k'' \right), \tag{2.5}$$

while varying with respect to the Lagrange multiplier Λ gives the constraint equation $\sum_{i=1}^{3} r_i^2 = 1$.

Now, assuming this constraint is satisfied, the equations of motion (2.4) and (2.5) are equivalent to

$$r_j r_i'' - r_i r_j'' = 2 r_i r_j \left(\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i \right) + r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k' \right)^2 - r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k' \right)^2, \tag{2.6}$$

$$\begin{aligned} \dot{r}_i r_j + r_i \dot{r}_j = {} & r_j \sum_{k=1}^{3} r_k (r_i r_k)' \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k' \right) + r_i \sum_{k=1}^{3} r_k (r_j r_k)' \left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k' \right) + \\ & + \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i'' - \tilde{\tilde{\phi}}_k'' \right) + \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k'' \right). \end{aligned} \tag{2.7}$$

Notice that the constraint term cancels out of equation (2.6).

2.2 Lax pair in the undeformed background

A Lax pair for this undeformed system [15], which is a function of the spectral parameter x, is

$$\mathcal{D}_\mu = \partial_\mu - A_\mu \qquad \text{with} \quad \mu = 0, 1, \tag{2.8}$$

where

$$A_0 = \frac{1}{6} [N, \partial_1 N] x + \frac{3i}{2} N x^2, \tag{2.9}$$

$$A_1 = i N x \tag{2.10}$$

and we have defined $N_{ij} = 3 U_i^* U_j - \delta_{ij}$, where $U_i = r_i e^{i \tilde{\tilde{\phi}}_i}$ and $\sum_{i=1}^{3} r_i^2 = 1$.

This satisfies the zero curvature condition $[\mathcal{D}_0, \mathcal{D}_1] = 0$, which is equivalent to

$$\partial_0 A_1 - \partial_1 A_0 - [A_0, A_1] = 0. \tag{2.11}$$

The above equation results in the Landau-Lifshitz equation of motion [15, 23] given by

$$i \partial_0 N = \frac{1}{6} [N, \partial_1^2 N] \tag{2.12}$$

and, upon substitution of $N_{ij} = 3 U_i^* U_j - \delta_{ij} = 3 r_i r_j e^{i(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i)} - \delta_{ij}$ into this equation, one obtains equations (2.6) and (2.7), which are equivalent to the undeformed equations of motion (see appendix A).

In terms of r_i and $\tilde{\tilde{\phi}}_i$, the undeformed Lax pair is $\mathcal{D}_\mu = \partial_\mu - A_\mu$, where

$$(A_\mu)_{ij} = (B_\mu)_{ij} \; e^{i(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i)} \qquad \text{with} \qquad \mu = 0, 1 \tag{2.13}$$

JHEP01(2006)050

and we define

$$(B_0)_{ij} = \left[\frac{3}{2}(r_i r_j' - r_i' r_j) + \frac{3i}{2} r_i r_j (\tilde{\tilde{\phi}}_i' + \tilde{\tilde{\phi}}_j') - 3i r_i r_j \sum_{k=1}^{3} r_k^2 \tilde{\tilde{\phi}}_k'\right] x + \frac{3i}{2}(3r_i r_j - \delta_{ij})\, x^2\,,$$
$$(B_1)_{ij} = i(3r_i r_j - \delta_{ij})\, x\,. \tag{2.14}$$

3. Strings in the γ_i-deformed background

3.1 Equations of motion in the γ_i-deformed background

We now generalize the previous results to strings in the γ_i-deformed $\mathbb{R} \times S^5$ background, which are again rotating in three orthogonal planes. The lagrangian for this γ_i-deformed background in the fast motion limit (derived in [15]) is

$$\mathcal{L} = \sum_{i=1}^{3} r_i^2 \dot{\phi}_i - \frac{\tilde{\lambda}}{2} \times \tag{3.1}$$
$$\times \left\{ \sum_{i=1}^{3} (r_i')^2 + \sum_{\substack{i,j=1\\ i<j}}^{3} r_i^2 r_j^2 \left(\phi_i' - \phi_j' - \sum_{k=1}^{3} \epsilon_{ijk} \bar{\gamma}_k\right)^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2 + \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1\right)\right\},$$

where $\bar{\gamma}_i = \tilde{\gamma}_i \mathcal{J} = \gamma_i J$ and $\bar{\gamma} = \sum_{i=1}^{3} \bar{\gamma}_i$.

This γ_i-deformed lagrangian (again re-defining $\tau \to \frac{1}{\tilde{\lambda}}\tau$ and $\mathcal{L} \to -\frac{1}{\tilde{\lambda}}\mathcal{L}$) can also be written, using the constraint $\sum_{i=1}^{3} r_i^2 = 1$, as

$$\mathcal{L} = -\sum_{i=1}^{3} r_i^2 \dot{\phi}_i + \frac{1}{2}\sum_{i=1}^{3} (r_i')^2 + \frac{1}{2} \times$$
$$\times \sum_{\substack{i,j=1\\ i<j}}^{3} r_i^2 r_j^2 \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m^2\right) - \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm} \bar{\gamma}_l r_m^2\right)\right]^2 +$$
$$+ \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} r_i^2 - 1\right). \tag{3.2}$$

Varying the above lagrangian with respect to r_i and ϕ_i and then using the constraint equation $\sum_{i=1}^{3} r_i^2 = 1$, which can be obtained by varying with respect to Λ, gives the γ_i-deformed equations of motion

$$r_i'' = -2r_i \Bigg\{ \dot{\phi}_i + \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm} \bar{\gamma}_m) -$$
$$- \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm} \bar{\gamma}_i) \Bigg\} +$$

$$
\begin{aligned}
&+r_i\sum_{k=1}^{3} r_k^2\left[\left(\phi_i'+\sum_{l,m=1}^{3}\epsilon_{ilm}\bar\gamma_l r_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m^2\right)\right]^2+\Lambda r_i\,, && (3.3)\\
\dot r_i =& \sum_{k=1}^{3} r_k(r_ir_k)'\left[\left(\phi_i'+\sum_{l,m=1}^{3}\epsilon_{ilm}\bar\gamma_l r_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m^2\right)\right]+\\
&+\frac{1}{2}\,r_i\sum_{k=1}^{3} r_k^2\left[\left(\phi_i''+2\sum_{l,m=1}^{3}\epsilon_{ilm}\bar\gamma_l r_m r_m'\right)-\left(\phi_k''+2\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m r_m'\right)\right]. && (3.4)
\end{aligned}
$$

Now, assuming this constraint is satisfied, the above equations of motion (3.3) and (3.4) are equivalent to

$$
\begin{aligned}
r_jr_i''-r_ir_j'' =\;& 2r_ir_j(\dot\phi_j-\dot\phi_i)-2r_ir_j\Bigg\{\sum_{l,m=1}^{3}\epsilon_{ilm}r_i^2r_l^2\bar\gamma_m(\phi_i'-\phi_l'-\epsilon_{ilm}\bar\gamma_m)\\
&-\frac{1}{2}\sum_{l,m=1}^{3}\epsilon_{ilm}r_l^2r_m^2(\bar\gamma_l+\bar\gamma_m)(\phi_l'-\phi_m'-\epsilon_{ilm}\bar\gamma_i)\Bigg\}+\\
&+2r_ir_j\Bigg\{\sum_{l,m=1}^{3}\epsilon_{jlm}r_j^2r_l^2\bar\gamma_m(\phi_j'-\phi_l'-\epsilon_{jlm}\bar\gamma_m)-\\
&-\frac{1}{2}\sum_{l,m=1}^{3}\epsilon_{jlm}r_l^2r_m^2(\bar\gamma_l+\bar\gamma_m)(\phi_l'-\phi_m'-\epsilon_{jlm}\bar\gamma_j)\Bigg\}+\\
&+r_ir_j\sum_{k=1}^{3} r_k^2\left[\left(\phi_i'+\sum_{l,m=1}^{3}\epsilon_{ilm}\bar\gamma_l r_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m^2\right)\right]^2-\\
&-r_ir_j\sum_{k=1}^{3} r_k^2\left[\left(\phi_j'+\sum_{l,m=1}^{3}\epsilon_{jlm}\bar\gamma_l r_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m^2\right)\right]^2, && (3.5)\\
\dot r_ir_j+r_i\dot r_j =\;& r_j\sum_{k=1}^{3} r_k(r_ir_k)'\left[\left(\phi_i'+\sum_{l,m=1}^{3}\epsilon_{ilm}\bar\gamma_l r_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m^2\right)\right]+ && (3.6)\\
&+r_i\sum_{k=1}^{3} r_k(r_jr_k)'\left[\left(\phi_j'+\sum_{l,m=1}^{3}\epsilon_{jlm}\bar\gamma_l r_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m^2\right)\right]+\\
&+\frac{1}{2}r_ir_j\sum_{k=1}^{3} r_k^2\left[\left(\phi_i''+2\sum_{l,m=1}^{3}\epsilon_{ilm}\bar\gamma_l r_m r_m'\right)-\left(\phi_k''+2\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m r_m'\right)\right]+\\
&+\frac{1}{2}r_ir_j\sum_{k=1}^{3} r_k^2\left[\left(\phi_j''+2\sum_{l,m=1}^{3}\epsilon_{jlm}\bar\gamma_l r_m r_m'\right)-\left(\phi_k''+2\sum_{l,m=1}^{3}\epsilon_{klm}\bar\gamma_l r_m r_m'\right)\right].
\end{aligned}
$$

3.2 Transformation from the undeformed equations of motion to the γ_i-deformed equations of motion

The transformation which takes the undeformed equations of motion into the γ_i-deformed

JHEP01(2006)050

equations of motion is

$$\dot{\tilde{\phi}}_i = \dot{\phi}_i + \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm}\bar{\gamma}_m) - \frac{1}{2}\sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm}\bar{\gamma}_i)\,,$$

$$\tilde{\phi}_i' = \phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\,. \tag{3.7}$$

Now, for this transformation to be valid, we must have $(\dot{\tilde{\phi}}_i)' = (\tilde{\phi}_i')^{\cdot}$. Thus the compatibility condition, which must be satisfied, is

$$2\sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m \dot{r}_m = \partial_1 \Bigg\{ \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm}\bar{\gamma}_m) - \\ - \frac{1}{2}\sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm}\bar{\gamma}_i) \Bigg\}\,. \tag{3.8}$$

However, from the equation of motion (3.4), we know that

$$r_i \dot{r}_i = \frac{1}{2}\,\partial_1 \left\{ \sum_{k=1}^{3} r_i^2 r_k^2 \left[\left(\phi_i' + \sum_{n,s=1}^{3} \epsilon_{ins}\bar{\gamma}_n r_s^2 \right) - \left(\phi_k' + \sum_{n,s=1}^{3} \epsilon_{kns}\bar{\gamma}_n r_s^2 \right) \right] \right\} \tag{3.9}$$

and thus

$$2\sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m \dot{r}_m = \tag{3.10}$$
$$= \partial_1 \left\{ \sum_{k,l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2 r_k^2 \left[\left(\phi_m' + \sum_{n,s=1}^{3} \epsilon_{mns}\bar{\gamma}_n r_s^2 \right) - \left(\phi_k' + \sum_{n,s=1}^{3} \epsilon_{kns}\bar{\gamma}_n r_s^2 \right) \right] \right\}.$$

By setting $i = 1, 2$ and 3, and evaluating equations (3.8) and (3.10) separately (see appendix B), these equations can be shown to be the same. Thus the compatibility condition is automatically satisfied if the γ_i-deformed equations of motion (and the constraint equation) are valid.

3.3 Lax pair in the γ_i-deformed background

The γ_i-deformed Lax pair shall now be derived from the undeformed one following a similar procedure to that discussed in [12].

First the $\tilde{\phi}_i$-dependence of the undeformed Lax pair will be gauged away. The zero curvature condition is $[\mathcal{D}_0, \mathcal{D}_1] = 0$, where $\mathcal{D}_\mu = \partial_\mu - A_\mu$ with $\mu = 0, 1$. This is equivalent to $[M\mathcal{D}_0 M^{-1}, M\mathcal{D}_1 M^{-1}] = 0$, for any invertible matrix M, so we can change

$$\mathcal{D}_\mu \to \tilde{\mathcal{D}}_\mu = M\mathcal{D}_\mu M^{-1} = \partial_\mu + M\partial_\mu M^{-1} - MA_\mu M^{-1}\,. \tag{3.11}$$

Thus an equivalent gauged Lax pair is

$$\tilde{\mathcal{D}}_\mu = \partial_\mu - \mathcal{R}_\mu\,, \qquad \text{where} \quad \mathcal{R}_\mu = MA_\mu M^{-1} - M\partial_\mu M^{-1}\,. \tag{3.12}$$

We shall take $M_{ij} = ie^{i\tilde{\tilde{\phi}}_i}\delta_{ij}$ and thus $M_{ij}^{-1} = -ie^{-i\tilde{\tilde{\phi}}_i}\delta_{ij}$. Therefore it follows from equation (2.13) that the gauged undeformed Lax pair is $\tilde{\mathcal{D}}_\mu = \partial_\mu - \mathcal{R}_\mu$, where

$$(\mathcal{R}_\mu)_{ij} = (B_\mu)_{ij} + i\partial_\mu \tilde{\tilde{\phi}}_i\ \delta_{ij} \tag{3.13}$$

and thus, using the definition of $(B_\mu)_{ij}$, we obtain

$$(\mathcal{R}_0)_{ij} = \left[\frac{3}{2}\left(r_i r_j' - r_i' r_j\right) + \frac{3i}{2} r_i r_j \left(\tilde{\tilde{\phi}}_i' + \tilde{\tilde{\phi}}_j'\right) - 3i r_i r_j \sum_{k=1}^{3} r_k^2 \tilde{\tilde{\phi}}_k'\right] x + \frac{3i}{2}(3r_i r_j - \delta_{ij})\ x^2 + i\dot{\tilde{\tilde{\phi}}}_i\ \delta_{ij}\,, \tag{3.14}$$

$$(\mathcal{R}_1)_{ij} = i(3r_i r_j - \delta_{ij})\ x + i\tilde{\tilde{\phi}}_i'\ \delta_{ij}\,. \tag{3.15}$$

We can now make use of the transformation (3.7) to obtain the gauged γ_i-deformed Lax pair $\tilde{\mathcal{D}}_\mu^{\gamma_i} = \partial_\mu - \mathcal{R}_\mu^{\gamma_i}$, where

$$\begin{aligned}(\mathcal{R}_0^{\gamma_i})_{ij} = {} & \frac{3}{2}(r_i r_j' - r_i' r_j) x + \frac{3i}{2} r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) + \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] x - \\ & -3i r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) x + \frac{3i}{2}\left(3r_i r_j - \delta_{ij}\right)\ x^2 + \\ & + i\Bigg\{\dot{\phi}_i + \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm}\bar{\gamma}_m) - \\ & \quad - \frac{1}{2}\sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm}\bar{\gamma}_i)\Bigg\}\ \delta_{ij}\,, \end{aligned} \tag{3.16}$$

$$(\mathcal{R}_1^{\gamma_i})_{ij} = i(3r_i r_j - \delta_{ij})\ x + i\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right)\ \delta_{ij}\,. \tag{3.17}$$

The zero curvature condition $[\tilde{\mathcal{D}}_0^{\gamma_i}, \tilde{\mathcal{D}}_1^{\gamma_i}] = 0$ is equivalent to

$$\partial_0 \mathcal{R}_1^{\gamma_i} - \partial_1 \mathcal{R}_0^{\gamma_i} - [\mathcal{R}_0^{\gamma_i}, \mathcal{R}_1^{\gamma_i}] = 0 \tag{3.18}$$

and the equations thus obtained from this gauged γ_i-deformed Lax pair (see appendix C) are equations (3.5) and (3.6), which are equivalent to the γ_i-deformed equations of motion, and the compatibility condition, which follows directly from these equations motion.

4. Conclusion

In this paper we have considered strings in $\mathbb{R} \times S^5$ (at the center of AdS_5) rotating in three orthogonal planes in the non-supersymmetric γ_i-deformed background, which was constructed in [12] using a series of three TsT-transformations. Our starting point has been the string action in the fast motion limit, in which the total angular momentum $J = J_1 + J_2 + J_3$ is large and we consider the leading order in $\tilde{\lambda} = \frac{\lambda}{J^2}$ (derived in [15]). This

JHEP01(2006)050

action is equivalent on the gauge theory side to the effective action of the corresponding γ_i-deformed spin chain in the continuum limit, in which the length of the spin chain becomes large [15].

We have first reviewed the construction in [15, 23] of a Lax pair in the undeformed case. It was then demonstrated that there exists a consistent transformation which takes the undeformed equations of motion into the γ_i-deformed equations of motion. This was used to construct a Lax pair describing rotating strings in the the γ_i-deformed background. Thus it was shown that the γ_i-deformed theory remains integrable in the fast motion limit.

A related topic for further investigation would be to attempt to calculate conserved quantities which follow from this Lax pair. Specifically one could try to construct the monodromy matrix and thus the quasi-momenta as a function of the spectral parameter for the undeformed and γ_i-deformed theories in the fast motion limit. Another interesting point of discussion is the physical meaning of the transformation which was used to construct the γ_i-deformed Lax pair.

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A. Derivation of the equations of motion from the Lax pair in the undeformed background

A.1 Derivation of the Landau-Lifshitz equation

The Lax pair is $\mathcal{D}_\mu = \partial_\mu - A_\mu$ [15], where

$$A_0 = \frac{1}{6}[N, \partial_1 N]x + \frac{3i}{2}Nx^2 \,, \tag{A.1}$$

$$A_1 = iNx \,. \tag{A.2}$$

The equation which must be satisfied is

$$\partial_0 A_1 - \partial_1 A_0 - [A_0, A_1] = 0 \,. \tag{A.3}$$

Note also that N satisfies the constraints $\mathrm{Tr}(N) = 0$ and $N^2 = N + 2$ [15, 23], due to the definition $N_{ij} = 3U_i^* U_j - \delta_{ij}$, where $U_i = r_i e^{i\tilde{\tilde{\phi}}_i}$, and the constraint $\sum_{i=1}^{3} r_i^2 = 1$.

Equation (A.3), written in terms of N, is

$$i\partial_0 N x - \frac{1}{6}\partial_1[N, \partial_1 N]x - \frac{3i}{2}\partial_1 N x^2 - \frac{i}{6}[[N, \partial_1 N], N]x^2 = 0 \tag{A.4}$$

and thus, equating different orders in x,

$$i\partial_0 N = \frac{1}{6}\partial_1[N, \partial_1 N] \,, \tag{A.5}$$

JHEP01(2006)050

$$\frac{3}{2}\partial_1 N = -\frac{1}{6}[[N, \partial_1 N], N]\,. \tag{A.6}$$

Equation (A.6) follows from the constraint $N^2 = N+2$ and equation (A.5) is equivalent to the Landau-Lifshitz equation of motion [15, 23] given by

$$i\partial_0 N = \frac{1}{6}[N, \partial_1^2 N]\,. \tag{A.7}$$

A.2 Derivation of the undeformed equations of motion in terms of r_i and $\tilde{\tilde{\phi}}_i$ from the Landau-Lifshitz equation

The definition of N in component form is $N_{ij} = 3r_i r_j e^{i(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i)} - \delta_{ij}$. Thus we obtain

$$\partial_0 N_{ij} = 3\left[(\dot{r}_i r_j + r_i \dot{r}_j) + i r_i r_j(\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i)\right]\; e^{i(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i)}\,, \tag{A.8}$$

$$\partial_1 N_{ij} = 3\left[(r_i' r_j + r_i r_j') + i r_i r_j(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_i')\right]\; e^{i(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i)} \tag{A.9}$$

and hence

$$\partial_1^2 N_{ij} = 3[(r_i'' r_j + 2r_i' r_j' + r_i r_j'') + 2i(r_i r_j)'(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_i') + i r_i r_j(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_i'') - r_i r_j(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_i')^2]\; e^{i(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i)}\,, \tag{A.10}$$

from which it follows that

$$[N, \partial_1^2 N]_{ij} = \sum_{k=1}^{3} N_{ik}\partial_1^2 N_{kj} - \sum_{k=1}^{3} \partial_1^2 N_{ik} N_{kj} \tag{A.11}$$

$$\begin{aligned} &= 9\Bigg\{[r_i r_j'' - r_j r_i''] + 2i\left[r_i \sum_{k=1}^{3} r_k (r_k r_j)'(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k') - r_j \sum_{k=1}^{3} r_k (r_i r_k)'(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i')\right] + \\ &\quad + i\left[r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k'') - r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_k'' - \tilde{\tilde{\phi}}_i'')\right] - \\ &\quad - \left[r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k')^2 - r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i')^2\right]\Bigg\}\; e^{i(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i)}\,. \end{aligned} \tag{A.12}$$

Now we use the Landau-Lifshitz equation (A.7) to obtain

$$\begin{aligned} &i(\dot{r}_i r_j + r_i \dot{r}_j) - r_i r_j(\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i) = \\ &\quad = \frac{1}{2}\Bigg\{[r_i r_j'' - r_j r_i''] + 2i\left[r_i \sum_{k=1}^{3} r_k (r_k r_j)'(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k') - r_j \sum_{k=1}^{3} r_k (r_i r_k)'(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i')\right] + \\ &\qquad + i\left[r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k'') - r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_k'' - \tilde{\tilde{\phi}}_i'')\right] - \\ &\qquad - \left[r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k')^2 - r_i r_j \sum_{k=1}^{3} r_k^2(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i')^2\right]\Bigg\}\,. \end{aligned} \tag{A.13}$$

Therefore, equating the real and imaginary parts of the above equation, we find that

$$\text{Re:}\quad r_j r_i'' - r_i r_j'' = 2 r_i r_j (\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i) + r_i r_j \sum_{k=1}^{3} r_k^2 (\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k')^2 - r_i r_j \sum_{k=1}^{3} r_k^2 (\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k')^2 \,,$$

$$\text{Im:}\quad \dot{r}_i r_j + r_i \dot{r}_j = r_j \sum_{k=1}^{3} r_k (r_i r_k)' (\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k') + r_i \sum_{k=1}^{3} r_k (r_j r_k)' (\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k') + + \frac{1}{2}\, r_i r_j \sum_{k=1}^{3} r_k^2 (\tilde{\tilde{\phi}}_i'' - \tilde{\tilde{\phi}}_k'') + \frac{1}{2}\, r_i r_j \sum_{k=1}^{3} r_k^2 (\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k'') \,, \qquad \text{(A.14)}$$

which can be compared with equations (2.6) and (2.7), and thus seen to be equivalent to the undeformed equations of motion.

B. Compatibility condition for the transformation from the undeformed equations of motion to the γ_i-deformed equations of motion

The compatibility condition for the transformation is

$$2 \sum_{l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m \dot{r}_m = \partial_1 \Big\{ \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm} \bar{\gamma}_m) - - \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm} \bar{\gamma}_i) \Big\} . \qquad \text{(B.1)}$$

The γ_i-deformed equation of motion (3.4) gives

$$2 \sum_{l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m \dot{r}_m = = \partial_1 \Big\{ \sum_{k,l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m^2 r_k^2 \Big[\Big(\phi_m' + \sum_{n,s=1}^{3} \epsilon_{mns} \bar{\gamma}_n r_s^2 \Big) - \Big(\phi_k' + \sum_{n,s=1}^{3} \epsilon_{kns} \bar{\gamma}_n r_s^2 \Big) \Big] \Big\} . \qquad \text{(B.2)}$$

First we shall evaulate

$$\{1\} \equiv \Big\{ \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm} \bar{\gamma}_m) - \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm} \bar{\gamma}_i) \Big\} \qquad \text{(B.3)}$$

for $i = 1, 2$ and 3 as follows:

$$\{1\}_{i=1} = r_1^2 r_2^2 \bar{\gamma}_3 (\phi_1' - \phi_2' - \bar{\gamma}_3) + r_1^2 r_3^2 \bar{\gamma}_2 (\phi_3' - \phi_1' - \bar{\gamma}_2) - r_2^2 r_3^2 (\bar{\gamma}_2 + \bar{\gamma}_3)(\phi_2' - \phi_3' - \bar{\gamma}_1), \qquad \text{(B.4)}$$

$$\{1\}_{i=2} = r_1^2 r_2^2 \bar{\gamma}_3 (\phi_1' - \phi_2' - \bar{\gamma}_3) + r_2^2 r_3^2 \bar{\gamma}_1 (\phi_2' - \phi_3' - \bar{\gamma}_1) - r_1^2 r_3^2 (\bar{\gamma}_1 + \bar{\gamma}_3)(\phi_3' - \phi_1' - \bar{\gamma}_2), \qquad \text{(B.5)}$$

$$\{1\}_{i=3} = r_1^2 r_3^2 \bar{\gamma}_2 (\phi_3' - \phi_1' - \bar{\gamma}_2) + r_2^2 r_3^2 \bar{\gamma}_1 (\phi_2' - \phi_3' - \bar{\gamma}_1) - r_1^2 r_2^2 (\bar{\gamma}_1 + \bar{\gamma}_2)(\phi_1' - \phi_2' - \bar{\gamma}_3). \qquad \text{(B.6)}$$

Now we shall determine

$$\{2\} \equiv \Big\{ \sum_{k,l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m^2 r_k^2 \Big[\Big(\phi_m' + \sum_{n,s=1}^{3} \epsilon_{mns} \bar{\gamma}_n r_s^2 \Big) - \Big(\phi_k' + \sum_{n,s=1}^{3} \epsilon_{kns} \bar{\gamma}_n r_s^2 \Big) \Big] \Big\} \qquad \text{(B.7)}$$

for $i = 1, 2$ and 3 as follows:

$$\begin{aligned}
\{2\}_{i=1} &= \bar\gamma_2 r_3^2(\phi_3' + \bar\gamma_1 r_2^2 - \bar\gamma_2 r_1^2) - \bar\gamma_3 r_2^2(\phi_2' + \bar\gamma_3 r_1^2 - \bar\gamma_1 r_3^2) - \bar\gamma_2 r_3^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') + \\
&\quad + \bar\gamma_3 r_2^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3 \\
&= \bar\gamma_2 r_3^2[(r_1^2 + r_2^2 + r_3^2)\phi_3' + \bar\gamma_1 r_2^2 - \bar\gamma_2 r_1^2] - \bar\gamma_3 r_2^2[(r_1^2 + r_2^2 + r_3^2)\phi_2' + \bar\gamma_3 r_1^2 - \bar\gamma_1 r_3^2] - \\
&\quad - \bar\gamma_2 r_3^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') + \bar\gamma_3 r_2^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') \qquad \text{(B.8)} \\
&= r_1^2 r_2^2 \bar\gamma_3(\phi_1' - \phi_2' - \bar\gamma_3) + r_1^2 r_3^2 \bar\gamma_2(\phi_3' - \phi_1' - \bar\gamma_2) - r_2^2 r_3^2(\bar\gamma_2 + \bar\gamma_3)(\phi_2' - \phi_3' - \bar\gamma_1)\,, \\
\{2\}_{i=2} &= \bar\gamma_3 r_1^2(\phi_1' + \bar\gamma_2 r_3^2 - \bar\gamma_3 r_2^2) - \bar\gamma_1 r_3^2(\phi_3' + \bar\gamma_1 r_2^2 - \bar\gamma_2 r_1^2) - \bar\gamma_3 r_1^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') + \\
&\quad + \bar\gamma_1 r_3^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') \\
&= \bar\gamma_3 r_1^2[(r_1^2 + r_2^2 + r_3^2)\phi_1' + \bar\gamma_2 r_3^2 - \bar\gamma_3 r_2^2] - \bar\gamma_1 r_3^2[(r_1^2 + r_2^2 + r_3^2)\phi_3' + \bar\gamma_1 r_2^2 - \bar\gamma_2 r_1^2] - \\
&\quad - \bar\gamma_3 r_1^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') + \bar\gamma_1 r_3^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') \qquad \text{(B.9)} \\
&= r_1^2 r_2^2 \bar\gamma_3(\phi_1' - \phi_2' - \bar\gamma_3) + r_2^2 r_3^2 \bar\gamma_1(\phi_2' - \phi_3' - \bar\gamma_1) - r_1^2 r_3^2(\bar\gamma_1 + \bar\gamma_3)(\phi_3' - \phi_1' - \bar\gamma_2)\,, \\
\{2\}_{i=3} &= \bar\gamma_1 r_2^2(\phi_2' + \bar\gamma_3 r_1^2 - \bar\gamma_1 r_3^2) - \bar\gamma_2 r_1^2(\phi_1' + \bar\gamma_2 r_3^2 - \bar\gamma_3 r_2^2) - \bar\gamma_1 r_2^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') + \\
&\quad + \bar\gamma_2 r_1^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') \\
&= \bar\gamma_1 r_2^2[(r_1^2 + r_2^2 + r_3^2)\phi_2' + \bar\gamma_3 r_1^2 - \bar\gamma_1 r_3^2] - \bar\gamma_2 r_1^2[(r_1^2 + r_2^2 + r_3^2)\phi_1' + \bar\gamma_2 r_3^2 - \bar\gamma_3 r_2^2] - \\
&\quad - \bar\gamma_1 r_2^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') + \bar\gamma_2 r_1^2(r_1^2\phi_1' + r_2^2\phi_2' + r_3^2\phi_3') \qquad \text{(B.10)} \\
&= r_1^2 r_3^2 \bar\gamma_2(\phi_3' - \phi_1' - \bar\gamma_2) + r_2^2 r_3^2 \bar\gamma_1(\phi_2' - \phi_3' - \bar\gamma_1) - r_1^2 r_2^2(\bar\gamma_1 + \bar\gamma_2)(\phi_1' - \phi_2' - \bar\gamma_3)
\end{aligned}$$

using the constraint $\sum_{i=1}^{3} r_i^2 = 1$. We have thus show that $\{1\} = \{2\}$ and therefore the compatibility condition follows from the γ_i-deformed equations of motion.

C. Derivation of the equations of motion from the gauged Lax pair in the γ_i-deformed background

The gauged γ_i-deformed Lax pair is $\tilde{\mathcal{D}}_\mu^{\gamma_i} = \partial_\mu - \mathcal{R}_\mu^{\gamma_i}$, where

$$\begin{aligned}
(\mathcal{R}_0^{\gamma_i})_{ij} &= \frac{3}{2}(r_i r_j' - r_i' r_j)\; x + \frac{3i}{2} r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar\gamma_l r_m^2\right) + \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar\gamma_l r_m^2\right)\right] x - \\
&\quad -3i r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar\gamma_l r_m^2\right) x + \frac{3i}{2}(3 r_i r_j - \delta_{ij})\; x^2 + \\
&\quad + i\Bigg\{\dot\phi_i + \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar\gamma_m(\phi_i' - \phi_l' - \epsilon_{ilm}\bar\gamma_m) - \\
&\quad - \frac{1}{2}\sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2(\bar\gamma_l + \bar\gamma_m)(\phi_l' - \phi_m' - \epsilon_{ilm}\bar\gamma_i)\Bigg\}\; \delta_{ij}\,, \qquad \text{(C.1)}
\end{aligned}$$

$$(\mathcal{R}_1^{\gamma_i})_{ij} = i(3 r_i r_j - \delta_{ij})\; x + i\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar\gamma_l r_m^2\right) \delta_{ij}\,. \qquad \text{(C.2)}$$

The zero curvature condition

$$\partial_0 \mathcal{R}_1^{\gamma_i} - \partial_1 \mathcal{R}_0^{\gamma_i} - [\mathcal{R}_0^{\gamma_i}, \mathcal{R}_1^{\gamma_i}] = 0 \tag{C.3}$$

must now be satisfied.

We substitute equations (C.1) and (C.2) into this condition and equate different orders of the spectral parameter x as follows:

$\underline{O(x^0):}$ At zeroth order in the spectral parameter we obtain

$$\begin{aligned} &i\partial_0 \left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m^2 \right) \delta_{ij} - i\partial_1 \Big\{ \dot{\phi}_i + \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm} \bar{\gamma}_m) \\ &- \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm} \bar{\gamma}_i) \Big\} \delta_{ij} = 0 \end{aligned} \tag{C.4}$$

and therefore

$$\begin{aligned} \partial_0 \Big(\sum_{l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m^2 \Big) = \partial_1 \Big\{ &\sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm} \bar{\gamma}_m) \\ &- \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm} \bar{\gamma}_i) \Big\}. \end{aligned} \tag{C.5}$$

This is just the compatibility condition for the transformation from the undeformed equations of motion to the γ_i-deformed equations of motion.

$\underline{O(x^1):}$ At first order in the spectral parameter we find that

$$\begin{aligned} &3i(\dot{r}_i r_j + r_i \dot{r}_j) - \frac{3}{2}(r_i r_j'' - r_i'' r_j) - \\ &- \frac{3i}{2} \partial_1 \left\{ r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm} \bar{\gamma}_l r_m^2 \right) + \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm} \bar{\gamma}_l r_m^2 \right) \right] \right\} + \\ &+ 3i\partial_1 \left\{ r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm} \bar{\gamma}_l r_m^2 \right) \right\} + 3 r_i r_j (\dot{\phi}_i - \dot{\phi}_j) + \\ &+ 3 r_i r_j \Big\{ \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm} \bar{\gamma}_m) - \\ &- \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm} \bar{\gamma}_i) \Big\} - \\ &- 3 r_i r_j \Big\{ \sum_{l,m=1}^{3} \epsilon_{jlm} r_j^2 r_l^2 \bar{\gamma}_m (\phi_j' - \phi_l' - \epsilon_{jlm} \bar{\gamma}_m) - \\ &- \frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{jlm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{jlm} \bar{\gamma}_j) \Big\} + \end{aligned}$$

JHEP01(2006)050

$$+\frac{3i}{2}(r_i r_j' - r_i' r_j)\left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] -$$

$$-\frac{3}{2} r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) + \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] \times$$

$$\left[\left(\phi_i' + \sum_{n,s=1}^{3} \epsilon_{ins}\bar{\gamma}_n r_s^2\right) - \left(\phi_j' + \sum_{n,s=1}^{3} \epsilon_{jns}\bar{\gamma}_n r_s^2\right)\right] +$$

$$+ 3 r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) \times$$

$$\times \left[\left(\phi_i' + \sum_{n,s=1}^{3} \epsilon_{ins}\bar{\gamma}_n r_s^2\right) - \left(\phi_j' + \sum_{n,s=1}^{3} \epsilon_{jns}\bar{\gamma}_n r_s^2\right)\right] = 0\,. \tag{C.6}$$

Thus, equating the real and imaginary parts, we obtain

Re:

$$r_i'' r_j - r_i r_j'' = 2 r_i r_j (\dot{\phi}_j - \dot{\phi}_i) -$$

$$-2 r_i r_j \left\{ \sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m (\phi_i' - \phi_l' - \epsilon_{ilm}\bar{\gamma}_m) - \right.$$

$$\left. -\frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm}\bar{\gamma}_i) \right\} +$$

$$+2 r_i r_j \left\{ \sum_{l,m=1}^{3} \epsilon_{jlm} r_j^2 r_l^2 \bar{\gamma}_m (\phi_j' - \phi_l' - \epsilon_{jlm}\bar{\gamma}_m) - \right.$$

$$\left. -\frac{1}{2} \sum_{l,m=1}^{3} \epsilon_{jlm} r_l^2 r_m^2 (\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{jlm}\bar{\gamma}_j) \right\} +$$

$$+ r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right)^2 - \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)^2\right] -$$

$$-2 r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) \times$$

$$\times\left[\left(\phi_i' + \sum_{n,s=1}^{3} \epsilon_{ins}\bar{\gamma}_n r_s^2\right) - \left(\phi_j' + \sum_{n,s=1}^{3} \epsilon_{jns}\bar{\gamma}_n r_s^2\right)\right], \tag{C.7}$$

Im:

$$\dot{r}_i r_j + r_i \dot{r}_j = \frac{1}{2}\partial_1 \left\{ r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) + + \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right]\right\} -$$

$$-\partial_1 \left\{ r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right)\right\} -$$

$$-\frac{1}{2}(r_i r_j' - r_i' r_j)\left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right]. \tag{C.8}$$

Now equation (C.7) is equivalent to

$$\begin{aligned}
r_j r_i'' - r_i r_j'' = {} & 2r_i r_j(\dot{\phi}_j - \dot{\phi}_i) - \\
& -2r_i r_j\left\{\sum_{l,m=1}^{3} \epsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m(\phi_i' - \phi_l' - \epsilon_{ilm}\bar{\gamma}_m) - \right. \\
& \left. -\frac{1}{2}\sum_{l,m=1}^{3} \epsilon_{ilm} r_l^2 r_m^2(\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{ilm}\bar{\gamma}_i)\right\} + \\
& +2r_i r_j\left\{\sum_{l,m=1}^{3} \epsilon_{jlm} r_j^2 r_l^2 \bar{\gamma}_m(\phi_j' - \phi_l' - \epsilon_{jlm}\bar{\gamma}_m) - \right. \\
& \left. -\frac{1}{2}\sum_{l,m=1}^{3} \epsilon_{jlm} r_l^2 r_m^2(\bar{\gamma}_l + \bar{\gamma}_m)(\phi_l' - \phi_m' - \epsilon_{jlm}\bar{\gamma}_j)\right\} + \\
& +r_i r_j \sum_{k=1}^{3} r_k^2\left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right)\right]^2 - \\
& -r_i r_j \sum_{k=1}^{3} r_k^2\left[\left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right)\right]^2, \qquad \text{(C.9)}
\end{aligned}$$

which can be seen by multiplying out the last two squared terms and noting that $\sum_{i=1}^{3} r_i^2 = 1$.

Equation (C.8) can be written as

$$\begin{aligned}
\dot{r}_i r_j + r_i \dot{r}_j = {} & \frac{1}{2}\,(r_i r_j' + r_i' r_j)\left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) + \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] + \\
& +\frac{1}{2} r_i r_j\left[\left(\phi_i'' + 2\sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) + \left(\phi_j'' + 2\sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m r_m'\right)\right] - \\
& -(r_i r_j' + r_i' r_j)\sum_{k=1}^{3} r_k^2\left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) - \\
& -2r_i r_j \sum_{k=1}^{3} r_k r_k'\left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) - \\
& -r_i r_j \sum_{k=1}^{3} r_k^2\left(\phi_k'' + 2\sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m r_m'\right) - \\
& -\frac{1}{2}(r_i r_j' - r_i' r_j)\left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] \qquad \text{(C.10)}
\end{aligned}$$

$$
\begin{aligned}
&= r_i' r_j \left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) + r_i r_j' \left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right) - \\
&- 2 r_i r_j \sum_{k=1}^{3} r_k r_k' \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) - \\
&- r_i' r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) - r_i r_j' \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right) + \\
&+ \frac{1}{2} r_i r_j \left(\phi_i'' + 2\sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) \frac{1}{2} r_i r_j \left(\phi_j'' + 2\sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m r_m'\right) - \\
&- r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k'' + 2\sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m r_m'\right), \qquad \text{(C.11)}
\end{aligned}
$$

which implies, if one uses the constraint $\sum_{i=1}^{3} r_i^2 = 1$ and thus $\sum_{i=1}^{3} r_i r_i' = 0$, that

$$
\begin{aligned}
\dot{r}_i r_j + r_i \dot{r}_j &= r_j \sum_{k=1}^{3} r_k (r_i r_k)' \left[\left(\phi_i' + \sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right)\right] + \qquad \text{(C.12)} \\
&+ r_i \sum_{k=1}^{3} r_k (r_j r_k)' \left[\left(\phi_j' + \sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m^2\right)\right] + \\
&+ \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_i'' + 2\sum_{l,m=1}^{3} \epsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) - \left(\phi_k'' + 2\sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m r_m'\right)\right] + \\
&+ \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_j'' + 2\sum_{l,m=1}^{3} \epsilon_{jlm}\bar{\gamma}_l r_m r_m'\right) - \left(\phi_k'' + 2\sum_{l,m=1}^{3} \epsilon_{klm}\bar{\gamma}_l r_m r_m'\right)\right].
\end{aligned}
$$

Equations (C.9) and (C.12) are the same as equations (3.5) and (3.6), and are thus equivalent to the γ_i-deformed equations of motion.

$\underline{O(x^2)}$: At second order in the spectral parameter, one obtains an equation which is trivially satisfied, again using the constraint $\sum_{i=1}^{3} r_i^2 = 1$ and hence that $\sum_{i=1}^{3} r_i r_i' = 0$.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional $N = 1$ supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [hep-th/9503121].

[5] O. Lunin and J. Maldacena, *Deforming field theories with* U(1) × U(1) *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086].

[6] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge - string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192].

[7] V. Niarchos and N. Prezas, *Bmn operators for $N = 1$ superconformal Yang-Mills theories and associated string backgrounds*, *JHEP* **06** (2003) 015 [hep-th/0212111].

[8] R. de Mello Koch, J. Murugan, J. Smolic and M. Smolic, *Deformed pp-waves from the Lunin-Maldacena background*, *JHEP* **08** (2005) 072 [hep-th/0505227].

[9] T. Mateos, *Marginal deformation of $N = 4$ SYM and Penrose limits with continuum spectrum*, *JHEP* **08** (2005) 026 [hep-th/0505243].

[10] S.S. Pal, *β-deformations, potentials and kk modes*, *Phys. Rev.* **D 72** (2005) 065006 [hep-th/0505257].

[11] D.Z. Freedman and U. Gursoy, *Comments on the beta-deformed $N = 4$ SYM theory*, *JHEP* **11** (2005) 042 [hep-th/0506128].

[12] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [hep-th/0503201].

[13] J.G. Russo, *String spectrum of curved string backgrounds obtained by T- duality and shifts of polar angles*, *JHEP* **09** (2005) 031 [hep-th/0508125].

[14] R.C. Rashkov, K.S. Viswanathan and Y. Yang, *$Ts \ldots sT$ transformation on $AdS_5 \times S^5$ background*, hep-th/0509058.

[15] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 731** (2005) 1 [hep-th/0507021].

[16] R. de Mello Koch, N. Ives, J. Smolic and M. Smolic, *Unstable giants*, hep-th/0509007.

[17] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[18] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *JHEP* **06** (2002) 007 [hep-th/0204226].

[19] M. Kruczenski, *Spin chains and string theory*, *Phys. Rev. Lett.* **93** (2004) 161602 [hep-th/0311203].

[20] H. Dimov and R.C. Rashkov, *A note on spin chain/string duality*, *Int. J. Mod. Phys.* **A 20** (2005) 4337 [hep-th/0403121].

[21] M. Kruczenski, A.V. Ryzhov and A.A. Tseytlin, *Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains*, *Nucl. Phys.* **B 692** (2004) 3 [hep-th/0403120].

[22] R. Hernandez and E. Lopez, *The* SU(3) *spin chain sigma model and string theory*, *JHEP* **04** (2004) 052 [hep-th/0403139].

[23] J. Stefanski, B. and A.A. Tseytlin, *Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations*, *JHEP* **05** (2004) 042 [hep-th/0404133].

[24] M. Kruczenski and A.A. Tseytlin, *Semiclassical relativistic strings in S^5 and long coherent operators in $N = 4$ SYM theory*, *JHEP* **09** (2004) 038 [hep-th/0406189].

[25] N.P. Bobev, H. Dimov and R.C. Rashkov, *Semiclassical strings in Lunin-Maldacena background*, hep-th/0506063.